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Note

Homogeneous Boundary Conditions for Pressure in the Mac Method

A new formulation for the continuity equation in the Marker-and-Cell (MAC) method has been found which yields an explicit pressure distribution from the solution of a Poisson equation with homogeneous boundary conditions. This new formulation is also free of the ambiguities of the old MAC in the region of inflow/outflow boundaries and convex corners. These changes result in a substantial simplification to the program and an average of 30 % reduction in computation time.

The Simplified Marker-and-Cell (SMAC) method [1] and [2] was developed by Amsden and Harlow as a major improvement to the MAC method. In both of these methods, the Navier–Stokes equations for the local conservation of momentum are cast into finite difference form. The equation for conservation of mass is transformed to a Poisson equation for the pressure distribution in MAC and for part of the scalar portion of a vector velocity potential in SMAC. The primary advantages of SMAC over MAC are that the boundary conditions for the Poisson equation in SMAC are homogeneous and that there are no ambiguities in the region of inflow/outflow boundaries and corners. MAC retains the advantage that pressure field is found as a direct result of the solution of its associated Poisson equation. This pressure field is required when the forces and moments exerted on the container by the liquid are desired and when the onset of cavitation or nucleate boiling is to be found.

A third approach to satisfying conservation of mass is described in this paper. A Poisson equation for the pressure distribution is still used, but it is formulated with homogeneous boundary conditions, and it is also free of ambiguities near corners and inflow/outflow boundaries. Thus, the new method retains the separate advantages of both MAC and SMAC. The other advantages of SMAC described in [1] and [2] also accrue to the present method, including the simplified formulation of the Poisson equation.

The pressure equation for MAC [3] is the finite difference analog of

$$\nabla^2 \phi = Q, \tag{1}$$

where ϕ is the ratio of pressure to density and Q is calculated directly from the velocity field and must include correction terms in the vicinity of corners, inflow/outflow boundaries, and some other mesh boundary conditions. It is,

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in some instances, difficult to formulate Q in (1) in the precisely consistent manner necessary to maintain small dilatation in the fluid. In this paper, the finite difference analog to Eq. (1) is first derived in a manner which eliminates most of the difficulty of consistency between the continuity and momentum equations. Then it is shown that the continuity equation can be altered to utilize homogeneous boundary conditions for the pressure. Computations are thereby simplified, and another potential source of inconsistency is eliminated.

The continuity equation in rectangular coordinates is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = D, \tag{2}$$

where D is the dilatation. Following the recommendations for MAC [4], D is retained in (2) and the entire equation is differentiated with respect to time. Finite differencing the resulting equation yields

$$\frac{1}{\delta x}\left[\left(\frac{\partial u}{\partial t}\right)_{i+1j+\frac{1}{2}} - \left(\frac{\partial u}{\partial t}\right)_{ij+\frac{1}{2}}\right] + \frac{1}{\delta y}\left[\left(\frac{\partial v}{\partial t}\right)_{i+\frac{1}{2}j+1} - \left(\frac{\partial v}{\partial t}\right)_{i+\frac{1}{2}j}\right] = -\frac{D_{ij}}{\delta t}, \tag{3}$$

where D_{ij}^{n+1} has been set equal to zero.

The mesh and velocities are shown in Fig. 1. All of the information required to calculate $\partial u/\partial t$ and $\partial v/\partial t$ is known at the beginning of a computational cycle except

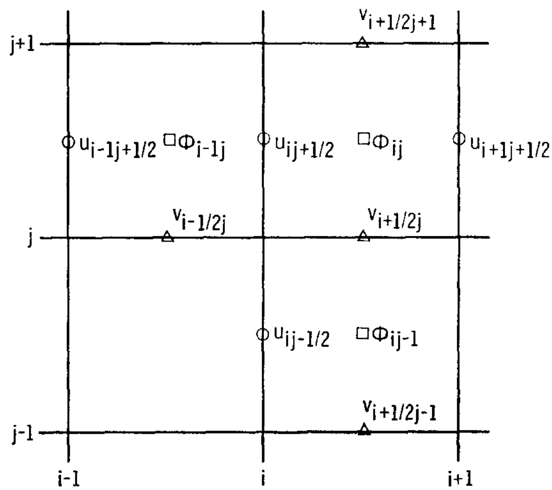

FIG. 1. Computational Mesh for MAC.

the pressure. Easton and Nelson [5] showed the utility of separating $\partial u/\partial t$ and $\partial v/\partial t$ into the known and unknown parts,

$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial t} - \frac{\partial \phi}{\partial x}, \qquad \text{and} \qquad \frac{\partial v}{\partial t} = \frac{\partial v'}{\partial t} - \frac{\partial \phi}{\partial y}. \tag{4}$$

Substituting into (3) yields (1) with Q defined in terms of $\partial u'/\partial t$ and $\partial v'/\partial t$. These quantities are already being calculated at some point in the computation cycle and their direct use in formulating Q_{ij} assures consistency between the momentum and continuity equations, except for cells adjacent to corners. The finite difference analog of (3) in terms of u', v', and ϕ, arranged for solution by iterative methods, is

$$\begin{aligned}\phi_{ij} = \frac{1}{\dfrac{2}{\delta x^2} + \dfrac{2}{\delta y^2}} \Bigg\{ & \frac{\phi_{i+1j} + \phi_{i-1j}}{\delta x^2} + \frac{\phi_{ij+1} + \phi_{ij-1}}{\delta y^2} \\ & - \frac{D_{ij}}{\delta t} - \frac{1}{\delta x}\left[\left(\frac{\partial u'}{\partial t}\right)_{i+1j+\frac{1}{2}} - \left(\frac{\partial u'}{\partial t}\right)_{ij+\frac{1}{2}}\right] \\ & - \frac{1}{\delta y}\left[\left(\frac{\partial v'}{\partial t}\right)_{i+\frac{1}{2}j+1} - \left(\frac{\partial v'}{\partial t}\right)_{i+\frac{1}{2}j}\right]\Bigg\}. \end{aligned} \tag{5}$$

Equation (5) is invariant with respect to changes in differencing technique, formulation of the Navier–Stokes equations, variable viscosity (as in turbulence), incorporation of buoyancy effects, and other changes which require reformulation of Q in MAC. Still present are the boundary inhomogenities and problems of consistency near corners and inflow/outflow boundaries.

The derivation of Eq. (5) can be modified so that the boundary conditions for the pressure are homogeneous. In MAC, solid wall boundary conditions for the pressure are obtained from the equation of momentum normal to the wall. This equation is solved for the pressure gradient by setting the time derivative of the normal velocity at the wall to zero. There results from (4)

$$\frac{\partial \phi}{\partial x} = \frac{\partial u'}{\partial t} \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = \frac{\partial v'}{\partial t} \tag{6}$$

on solid boundaries. As a specific example, let the left side of cell i, j be a solid boundary. The procedure in MAC for setting the pressure boundary condition is to difference (6), sometimes simplifying the right-hand side, but always, in essence, obtaining

$$\frac{\phi_{ij} - \phi_{i-1j}}{\delta x} = \left(\frac{\partial u'}{\partial t}\right)_{ij+\frac{1}{2}}. \tag{7}$$

Rather than formulate Eqs. (5) and (7) in finite difference form separately, the desired simplification results from eliminating $\partial u'/\partial t)_{ij+1/2}$ between these equations. Direct substitution from (7) into (5) yields the modified pressure equation,

$$\phi_{ij} = \frac{1}{\frac{2}{\delta x^2} + \frac{2}{\delta y^2}} \left\{ \frac{\phi_{i+1j} + \phi_{ij}}{\delta x^2} + \frac{\phi_{ij+1} - \phi_{ij-1}}{\delta y^2} - \frac{D_{ij}}{\delta t} - \frac{1}{\delta x}\left[\left(\frac{\partial u'}{\partial t}\right)_{i+1j+\frac{1}{2}} - \left(\frac{\partial u}{\partial t}\right)_{ij+\frac{1}{2}}\right] + \frac{1}{\delta y}\left[\left(\frac{\partial v'}{\partial t}\right)_{i+\frac{1}{2}j+1} - \left(\frac{\partial v'}{\partial t}\right)_{i+\frac{1}{2}j}\right]\right\}. \tag{8}$$

Note that ϕ_{ij} appears on the right-hand side of Eq. (8) in place of ϕ_{i-1j} ; therefore, the boundary condition is homogeneous.

Equation (8) is exactly the equation that would have been obtained had the known quantity $(\partial u/\partial t)_{ij+\frac{1}{2}}$ been retained in (3) instead of being removed by substituting from (4). It follows that, as written, (8) is equally valid for inflow or outflow boundaries when $(\partial u/\partial t)_{ij+\frac{1}{2}}$ is known and may be different from zero. Furthermore, the formulation (8) does not suffer from ambiguity or inconsistency near inflow/outflow boundaries and corners, so long as the velocities required to calculate $\partial u'/\partial t$ and $\partial v'/\partial t$ are chosen in the same manner everywhere they are used.

The new procedure is very easy to incorporate into a MAC program. On all boundaries, the actual, known time derivative of the normal velocity is used in (5) in place of $\partial u'/\partial t$ or $\partial v'/\partial t$ and the pressure/density ratio outside the boundary is replaced by ϕ_{ij}. This procedure works in the presence of other boundaries and when the equations are formulated in any other coordinate systems. The pressure and velocity fields computed are identical to those computed by the standard MAC method.

The number of computations required for each iteration cycle in the solution of the pressure equation has been substantially reduced. The resulting savings in execution time on the computer will vary with the problem solved and the detailed coding of the program. A 30% reduction in computer run time for a simulation of low gravity propellant reorientation flow resulted from these simplifications. Comparable reductions of run time have been noted for other cases involving rapidly changing pressure fields.

Conclusions

While the SMAC method represents a major improvement over MAC, the advantages which accrue to SMAC can be made available in a formally identical procedure without sacrificing the direct calculation of the pressure field. Therefore, the potential function employed in SMAC to satisfy continuity can be manipulated to yield the pressure field directly. The resultant program is substantially simpler than MAC, resulting in an important reduction in computation time, but more important is the fact that changes in the program may be made much more easily than before because of the much simpler logic that results.

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